LORETO KIRRIBILLI
MARIA REGINA ANGELORUM
CRUCI DUM SPIRO FIDO

2011

YEAR 11

HSC ASSESSMENT TASK 1

Mathematics Extension 2

Date: Monday 5th December

Weighting: 10% of the HSC assessment: 5% Concepts, Skills and Techniques
5% Reasoning and Communication

General Instructions

- Reading time – 2 minutes
- Working time – 45 minutes
- Write using blue or black pen
- Draw diagrams using pencil
- Board-approved calculators may be used
- All necessary working should be shown in every question

Total marks – 34

- Attempt all questions

Outcomes to be assessed:

E2 A student chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings

E3 A student uses the relationship between algebraic and geometric representations of complex numbers

E4 A student uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving polynomials

E9 A student communicates abstract ideas and relationships using appropriate notation and logical argument

Section I   Student Number: _____________

4 marks
Attempt Questions 1 – 4
Circle the correct response (A, B, C or D) on the questions below.

1. Which of the following represents $z\bar{z}$?

(A) $x^2 - y^2$ (B) $x^2 + y^2$ (C) 1 (D) $y^2 - x^2$

2. Which of the following represents $z + iw$, given $z = 3 - 4i$ and $w = 2 + i$.

(A) $1 - 5i$ (B) $2 - 2i$ (C) $4 - 2i$ (D) $5 - 3i$

3. Which of the following represents $\frac{z}{w}$, given $z = 2 + 3i$ and $w = 3 - 2i$.

(A) $\frac{12}{5} + \frac{13i}{5}$

(B) $\frac{12}{13} + i$

(C) $\frac{13i}{5}$

(D) i

4. What is the general solution of $\tan 4\theta = 1$?

(A) $\frac{\pi}{16} + \frac{n\pi}{4}$, where n is an integer.

(B) $\frac{\pi}{16} + \frac{n\pi}{2}$, where n is an integer.

(C) $\frac{\pi}{4} + \frac{n\pi}{4}$, where n is an integer.

(D) $\frac{\pi}{4} + \frac{n\pi}{2}$, where n is an integer.

Section II

30 marks
Attempt questions 5 – 6
Answer each question on the paper provided.
All necessary working should be shown on every question.

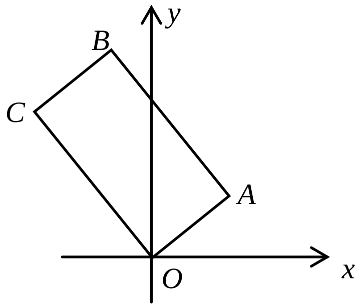
Question 5 (15 marks) **Marks**

(a) Let $z = 1 - i$.

(i) Write z in modulus-argument form. **1**

(ii) Hence show that $z^4 + 4 = 0$. **2**

(b)



OABC is a rectangle in the Argand diagram where *O* is the origin and the point *A* represents the complex number $2 + i$.

(i) Given that the length of the rectangle is three times its breadth, and *OA* is one of its shorter sides, find the complex numbers that correspond to *B* and *C*. **2**

(ii) This rectangle is rotated about *O* through 30°. Find the complex number that corresponds to the new position of *A*. **2**

(c) Consider the equation $z^3 + mz^2 + nz + 6 = 0$, where *m* and *n* are real. It is known that $1 - i$ is a root of the equation.

(i) Find the other two roots of the equation. **2**

(ii) Find the values of *m* and *n*. **2**

(d) If α, β and γ are the roots of the equation $x^3 - 2x^2 + 3x - 4 = 0$. Find:

(i) the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$. **2**

(ii) $\alpha^3 + \beta^3 + \gamma^3$. **2**

Question 6 (15 marks) START A NEW PAGE **Marks**

(a) (i) Show that $\sin x + \sin 3x = 2\sin 2x \cos x$. **1**

(ii) Hence, or otherwise, find all solutions of: **3**

$$\sin x + \sin 2x + \sin 3x = 0, \text{ for } 0 \le x \le 2\pi.$$

(b) (i) On an Argand diagram, sketch the region where the inequality $|z - 2i| \le 1$ is satisfied. **2**

(ii) Determine the maximum value of $|z|$ and $\arg z$. **2**

(c) β is the complex root of $z^5 - 1 = 0$, with the smallest positive argument.

(i) Show that the other complex roots of $z^5 - 1 = 0$ can be written as powers of β and plot all roots on an Argand diagram. **3**

(ii) Find the quadratic equation whose roots are $\beta + \beta^4$ and $\beta^2 + \beta^3$. **2**

(iii) Hence, or otherwise, deduce that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. **2**

End of Examination

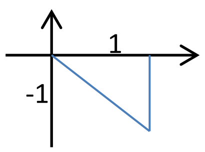
Year 11
Mathematics Extension 2
Term 4 Exam 2011
Solutions and Marking Criteria

Section I

1.	(B) $z\bar{z} = (x+iy)(x-iy)$ $= x^2 - i^2y^2$ $= x^2 + y^2$
2.	(B) $z + iw = 3 - 4i + i(2+i)$ $= 3 - 4i + 2i - 1$ $= 2 - 2i$
3.	(D) $\frac{z}{w} = \frac{2+3i}{3-2i} \times \frac{3+2i}{3+2i}$ $= \frac{6+4i+9i-6}{9+4}$ $= \frac{13i}{13}$ $= i$
4.	(A) $\tan 4\theta = 1$ $4\theta = \frac{\pi}{4} + n\pi$ $\theta = \frac{\pi}{16} + \frac{n\pi}{4}$, where n is an integer.

Section II

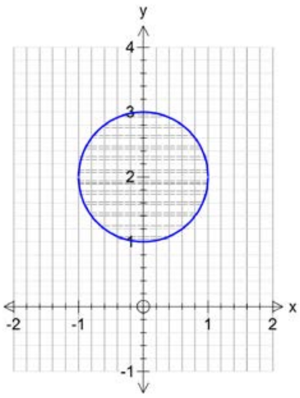
Question 5:

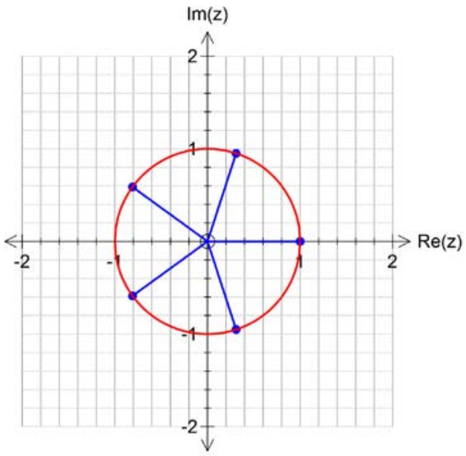
a i	 $z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$	1 – correct answer

a ii	$LHS = z^4 + 4$ $= \left(\sqrt{2}\right)^4 \left(\cos(-\pi) + i\sin(-\pi)\right) + 4$ $= 4(-1) + 4$ $= 0$ $= RHS$	2 – correct solution 1 – solution with one error
b i	$C = 3i(2+i)$ $= -3 + 6i$ $B = 2 + i - 3 + 6i$ $= -1 + 7i$	2 – correct solution 1 – correct complex number for *B* or *C* only
b ii	$(2+i)\left(\cos 30^\circ + i\sin 30^\circ\right)$ $= (2+i)\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$ $= \sqrt{3} + i + \frac{\sqrt{3}}{2}i - \frac{1}{2}$ $= \left(\sqrt{3} - \frac{1}{2}\right) + i\left(1 + \frac{\sqrt{3}}{2}\right)$	2 – correct solution 1 – correct attempt at solution
c i	$z^3 + mz^2 + nz + 6 = 0$ $1 - i$ is a root $\therefore 1 + i$ is also a root $\alpha(1+i)(1-i) = -6$ $\alpha(1+1) = -6$ $\alpha = -3$ Roots: $1 \pm i, -3$	2 – correct solution 1 – correctly finds one other root
c ii	$\alpha + \beta + \gamma = -\frac{b}{a}$ $1 - i + 1 + i - 3 = -m$ $m = 1$ $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$ $(1+i)(1-i) - 3(1+i) - 3(1-i) = n$ $1 + 1 - 3 + 3i - 3 - 3i = n$ $n = -4$	2 – correct solution 1 – correct value for *m* or *n*
d i	$x^3 - 2x^2 + 3x - 4 = 0$ $\left(\frac{1}{x}\right)^3 - 2\left(\frac{1}{x}\right)^2 + 3\left(\frac{1}{x}\right) - 4 = 0$ $\frac{1}{x^3} - \frac{2}{x^2} + \frac{3}{x} - 4 = 0$ $1 - 2x + 3x^2 - 4x^3 = 0$	2 – correct solution 1 – correct attempt at solution

d ii	α is a root of $x^3-2x^2+3x-4=0$ $\therefore \alpha^3-2\alpha^2+3\alpha-4=0$ $\alpha^3=2\alpha^2-3\alpha+4$ Similarly $\beta^3=2\beta^2-3\beta+4$ $\gamma^3=2\gamma^2-3\gamma+4$ $\alpha^3+\beta^3+\gamma^3$ $=2(\alpha^2+\beta^2+\gamma^2)-3(\alpha+\beta+\gamma)+12$ $=2\left[(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)\right]-3(\alpha+\beta+\gamma)+12$ $=2\left[2^2-2(3)\right]-3(2)+12$ $=-4-6+12$ $=2$	2 – correct solution 1 – finding an expression for $\alpha^3+\beta^3+\gamma^3$

Question 6:

a i	$LHS=\sin x+\sin 3x$ $=\sin(2x-x)+\sin(2x+x)$ $=\sin 2x\cos x-\cos 2x\sin x+\sin 2x\cos x+\cos 2x\sin x$ $=2\sin 2x\cos x$ $=RHS$	1 – correct solution
a ii	$\sin x+\sin 2x+\sin 3x=0$ $2\sin 2x\cos x+\sin 2x=0$ $\sin 2x(2\cos x+1)=0$ $\sin 2x=0$ $\quad\quad$ $\cos x=-\frac{1}{2}$ $2x=0,\pi,2\pi,3\pi,4\pi$ $\quad\quad$ $x=\frac{2\pi}{3},\frac{4\pi}{3}$ $x=0,\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi$ $\therefore x=0,\frac{\pi}{2},\frac{2\pi}{3},\pi,\frac{4\pi}{3},\frac{3\pi}{2},2\pi$	3 – correct solution 2 – solution with one error 1 – showing $\sin 2x(2\cos x+1)=0$
b i		2 – correct region 1 – region with one error

b ii	Max. value $\|z\|=3$ Max. value $\arg z=\frac{\pi}{2}+\frac{\pi}{6}$ $=\frac{2\pi}{3}$	2 – correct solution 1 – correct max. value of $\|z\|$ or arg z
c i	$z^5=1=\cos 0+i\sin 0$ Let $z=r(\cos\theta+i\sin\theta)$ be the roots $\therefore z^5=r^5(\cos 5\theta+i\sin 5\theta)$ $r^5=1 \qquad 5\theta=2k\pi \quad k=0,1,2,3,4$ $r=1 \qquad \theta=\frac{2k\pi}{5}$ $z_1=\cos 0+i\sin 0$ $z_2=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}=\beta$ $z_3=\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}$ $=\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^2$ $=\beta^2$ $z_4=\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}=\beta^3$ $z_5=\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}=\beta^4$ 	3 – correct solution 2 – correct roots but not plotted on an Argand diagram OR correct roots and plot but not in terms of powers of β. 1 – correct roots not in terms of β and not plotted on an Argand diagram

c ii	Eqn with roots $\beta+\beta^4,\ \beta^2+\beta^3$ Sum of roots = $\beta+\beta^4+\ \beta^2+\beta^3$ $=-1\left(\text{as sum of roots of } z^5-1=0 \text{ is zero}\right)$ Product of roots $=\left(\beta+\beta^4\right)\left(\beta^2+\beta^3\right)$ $=\beta^3+\beta^4+\beta^6+\beta^7$ $=\beta^3+\beta^4+\beta+\beta^2\ \ (\text{as } \beta^5=1)$ $=-1$ $\therefore\ \ x^2+x-1=0$	2 – correct solution 1 – progress towards correct solution
c iii	$\beta+\beta^4$ $=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}$ $=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\frac{-2\pi}{5}+i\sin\frac{-2\pi}{5}$ $=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}$ $=2\cos\frac{2\pi}{5}$ Similarly $\beta^2+\beta^3=2\cos\frac{4\pi}{5}$ $\beta+\beta^4+\beta^2+\beta^3=-1$ $2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}=-1$ $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$	2 – correct solution 1 – finding $\cos\frac{2\pi}{5}$ or $\cos\frac{4\pi}{5}$ in terms of β

Communication: (6 marks)

Question 5

b) i) 1 mark Communication of solution
b) ii) 1 mark Communication of solution
c) i) 1 mark Communication of solution

Question 6

a) i) 1 mark Communication of solution
c) i) 1 mark Communication of solution
c) ii) 1 mark Communication of solution